

Fuzzy sets in \leq -hypergroupoids

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Abstract. This paper serves as an example to show the way we pass from ordered groupoids (ordered semigroups) to ordered hypergroupoids (ordered hypersemigroups), from groupoids (semigroups) to hypergroupoids (hypersemigroups). The results on semigroups (or on ordered semigroup) can be transferred to hypersemigroups (or to ordered hypersemigroups) in the way indicated in the present paper.

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1 Introduction and prerequisites

An ordered groupoid ($:$ *po*-groupoid) is a nonempty set S endowed with an order “ \leq ” and a multiplication “ \cdot ” such that $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$ for every $c \in S$. Let (S, \cdot, \leq) be an ordered groupoid. A nonempty subset A of S is called a *left* (resp. *right*) ideal of S if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. A subset A of S is called an *ideal* of S if it is both a left and a right ideal of S [2]. A subgroupoid F of S is called a *filter* of S if (1) $a, b \in S$ such that $ab \in F$ implies $a \in F$ and $b \in F$ and (2) if $a \in F$ and $S \ni b \geq a$, then $b \in F$ [1].

Given a set S , a fuzzy subset of S (or a fuzzy set in S) is, by definition, an arbitrary mapping of S into the closed interval $[0, 1]$ of real numbers (Zadeh). Fuzzy sets in ordered groupoids have been first considered in [4], where the following concepts have been introduced and studied: A fuzzy subset f of an ordered groupoid (S, \cdot, \leq) is called a *fuzzy left* (resp. *fuzzy right*) *ideal* of S if (1) $x \leq y$ implies $f(x) \geq f(y)$ and (2) if $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$) for every $x, y \in S$. f is called a *fuzzy ideal* of S if it is both a fuzzy left ideal and a fuzzy right ideal of S . A fuzzy subset f of S is called a *fuzzy filter* of S if (1) $x \leq y$ implies $f(x) \leq f(y)$ and (2) if $f(xy) = \min\{f(x), f(y)\}$ for all $x, y \in S$. A fuzzy subset f of a groupoid S is called *prime* if $f(xy) \leq \max\{f(x), f(y)\}$ for all $x, y \in S$. For a groupoid S and a fuzzy subset f of S , the complement of f is

the fuzzy subset $f' : S \rightarrow [0, 1]$ of S defined by $f'(x) = 1 - f(x)$ for all $x \in S$. We have seen in [4] that a nonempty subset A of an ordered groupoid S is a left (resp. right) ideal of S if and only if its characteristic function f_A is a fuzzy left (resp. right) ideal of S . A nonempty subset F of an ordered groupoid S is a filter of S if and only if the fuzzy subset f_F is a fuzzy filter of S . A fuzzy subset f of an ordered groupoid S is a fuzzy filter of S if and only if the complement f' of f is a fuzzy prime ideal of S .

In the present paper we examine the results of ordered groupoids given in [4] for ordered hypergroupoids. We deal with an hypergroupoid (H, \circ) endowed with a relation “ \leq ” (not order relation, and so not compatible with the hyperoperation “ \circ ” in general). Though we could call σ that relation and σ -hypergroupoid the hypergroupoid endowed with the relation σ , we will show by “ \leq ” the relation and use the term \leq -hypergroupoid, to emphasize the fact that our results hold for ordered hypergroupoids as well. As a consequence, the results in [4] also hold in groupoids endowed with a relation “ \leq ” which is not an order in general. Our aim is to show the way we pass from ordered groupoids to ordered hypergroupoids.

For a groupoid $(S, .)$ we have one operation corresponding to each $(a, b) \in S \times S$ the unique element ab of S . For an hypergroupoid H we have two “operations”. One of them is the “operation” between the elements of H which is called hyperoperation as it maps the set $H \times H$ into the set of nonempty subsets of H and the other is the operation between the nonempty subsets of H . We use the terms left (right) ideal, bi-ideal, quasi-ideal instead of left (right) hyperideal, bi-hyperideal, quasi-hyperideal and so on, and this is because in this structure there are not two kind of left ideals, for example, to distinguish them as left ideal and left hyperideal. The left ideal in this structure is that one which corresponds to the left ideal of groupoids.

2 Main results

An *hypergroupoid* is a nonempty set H with an hyperoperation

$$\circ : H \times H \rightarrow \mathcal{P}^*(H) \mid (a, b) \rightarrow a \circ b$$

on H and an operation

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \rightarrow \mathcal{P}^*(H) \mid (A, B) \rightarrow A * B$$

on $\mathcal{P}^*(H)$ (induced by the operation of H) such that

$$A * B = \bigcup_{(a,b) \in A \times B} (a \circ b)$$

for every $A, B \in \mathcal{P}^*(H)$.

The operation “ $*$ ” is well defined. Indeed: If $(A, B) \in \mathcal{P}^*(H) \times \mathcal{P}^*(H)$, then $A * B = \bigcup_{(a,b) \in A \times B} (a \circ b)$. For every $(a, b) \in A \times B$, we have $(a, b) \in H \times H$, then $(a \circ b) \in \mathcal{P}^*(H)$, thus we get $A * B \in \mathcal{P}^*(H)$. If $(A, B), (C, D) \in \mathcal{P}^*(H) \times \mathcal{P}^*(H)$ such that $(A, B) = (C, D)$, then

$$A * B = \bigcup_{(a,b) \in A \times B} (a \circ b) = \bigcup_{(a,b) \in C \times D} (a \circ b) = C * D.$$

As the operation “ $*$ ” depends on the hyperoperation “ \circ ”, an hypergroupoid can be also denoted by (H, \circ) (instead of $(H, \circ, *)$).

If H is an hypergroupoid then, for every $x, y \in H$, we have

$$\{x\} * \{y\} = x \circ y.$$

Indeed, $\{x\} * \{y\} = \bigcup_{\substack{u \in \{x\} \\ v \in \{y\}}} (u \circ v) = x \circ y$.

The following proposition, though clear, plays an essential role in the theory of hypergroupoids.

Proposition 1. *Let (H, \circ) be an hypergroupoid, $x \in H$ and $A, B \in \mathcal{P}^*(H)$. Then we have the following:*

- (1) $x \in A * B \iff x \in a \circ b$ for some $a \in A, b \in B$.
- (2) If $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$.

A nonempty subset A of an hypergroupoid (H, \circ) is called a left (resp. right) ideal of H if $H * A \subseteq A$ (resp. $A * H \subseteq A$). A subset of H which is both a left ideal and a right ideal of H is called an *ideal* of H . A nonempty subset A of H is called a *subgroupoid* of H if $A * A \subseteq A$. Clearly, every left ideal, right ideal or ideal of H is a subgroupoid of H .

Definition 2. By a \leq -hypergroupoid we mean an hypergroupoid H endowed with a relation denoted by “ \leq ”.

We write $b \geq a$ if $a \leq b$.

Definition 3. Let H be a \leq -hypergroupoid. A fuzzy subset f of H is called a *fuzzy left ideal* of H if

- (1) $x \leq y \Rightarrow f(x) \geq f(y)$ and
- (2) if $f(x \circ y) \geq f(y)$ for all $x, y \in H$.

With the property (2) we mean the following:

- (2) if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(y)$.

A fuzzy subset f of H is called a *fuzzy right ideal* of H if

- (1) $x \leq y \Rightarrow f(x) \geq f(y)$ and
- (2) if $f(x \circ y) \geq f(x)$ for all $x, y \in H$.

With the property (2) we mean:

- (2) if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(x)$.

A fuzzy subset of H is called a *fuzzy ideal* of H if it is both a fuzzy left and a fuzzy right ideal of H . As one can easily see, a fuzzy subset f of H is a fuzzy ideal of H if and only

- (1) $x \leq y$ implies $f(x) \geq f(y)$ and
- (2) if $f(x \circ y) \geq \max\{f(x), f(y)\}$ for all $x, y \in H$ in the sense that

$$x, y \in H \text{ and } u \in x \circ y \text{ implies } f(u) \geq \max\{f(x), f(y)\}.$$

Following Zadeh, any mapping $f : H \rightarrow [0, 1]$ of a \leq -hypergroupoid H into the closed interval $[0, 1]$ of real numbers is called a *fuzzy subset* of H (or a *fuzzy set* in H) and f_A (: the characteristic function of A) is the mapping

$$f_A : H \rightarrow \{0, 1\} \mid x \rightarrow f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Definition 4. Let H be a \leq -hypergroupoid. A nonempty subset A of H is called a *left* (resp. *right*) *ideal* of H if

- (1) $H * A \subseteq A$ (resp. $A * H \subseteq A$) and
- (2) if $a \in A$ and $H \ni b \leq a$, then $b \in A$.

Lemma 5. Let (H, \circ) be an hypergroupoid. If A is a left (resp. right) ideal of H , then for every $h \in H$ and every $a \in A$, we have $h \circ a \subseteq A$ (resp. $a \circ h \subseteq A$).

“Conversely”, if A is a nonempty subset of H such that $h \circ a \subseteq A$ (resp. $a \circ h \subseteq A$) for every $h \in H$ and every $a \in A$, then the set A is a left (resp. right) ideal of H .

Proposition 6. Let H be a \leq -hypergroupoid. If L is a left ideal of H , then f_L is a fuzzy left ideal of H . “Conversely”, if L is a nonempty subset of H such that f_L is a fuzzy left ideal of H , then L is a left ideal of H .

Proof. \Rightarrow . Let L be a left ideal of H . By definition, f_L is a fuzzy subset of H . Let $x \leq y$. If $y \notin L$, then $f_L(y) = 0$, so $f_L(x) \geq f_L(y)$. If $y \in L$, then $H \ni x \leq y \in L$ and, since L is a left ideal of H , we have $x \in L$. Then $f_L(x) = f_L(y) = 1$, so $f_L(x) \geq f_L(y)$. Let now $x, y \in H$ and $u \in x \circ y$. Then $f_L(u) \geq f_L(y)$. Indeed: If $y \in L$ then, by Proposition 1(2), we have $x \circ y \subseteq H * L \subseteq L$, so $u \in L$, then $f_L(y) = f_L(u) = 1$. If $y \notin L$, then $f_L(y) = 0 \leq f_L(u)$.

\Leftarrow . Let $x \in H$ and $y \in L$. Then $x \circ y \subseteq L$. Indeed: Let $x \circ y \not\subseteq L$. Then there exists $u \in x \circ y$ such that $u \notin L$. Since $u \in x \circ y$, by hypothesis, we have $f_L(u) \geq f_L(y)$. Since $u \notin L$, we have $f_L(u) = 0$. Since $y \in L$, we have $f_L(y) = 1$, then $0 \geq 1$ which is impossible. Let now $x \in L$ and $H \ni y \leq x$. Then $y \in L$. Indeed: Since f_L is a fuzzy left ideal of H and $y \leq x$, we have $f_L(y) \geq f_L(x)$. Since $x \in L$, $f_L(x) = 1$. Then we have $f_L(y) \geq 1$. On the other hand, $f_L(y) \leq 1$, so we have $f_L(y) = 1$, and $y \in L$. By Lemma 5, L is a left ideal of H . \square

In a similar way we prove the following:

Proposition 7. Let H be a \leq -hypergroupoid. If R is a right ideal of H , then f_R is a fuzzy right ideal of H . “Conversely”, if R is a nonempty subset of H such that f_R is a fuzzy right ideal of H , then R is a right ideal of H .

Proposition 8. If H is a \leq -hypergroupoid, a nonempty subset I of H is an ideal of H if and only if f_I is a fuzzy ideal of H .

Now we introduce the concept of filters and fuzzy filters in \leq -hypergroupoids, and we characterize the filters of \leq -hypergroupoids in terms of fuzzy filters.

Definition 9. Let H be a \leq -hypergroupoid. A nonempty subset F of H is called a *filter* of H if

- (1) if $x, y \in F$, then $x \circ y \subseteq F$.
- (2) if $x, y \in H$ and $x \circ y \subseteq F$, then $x \in F$ and $y \in F$.
- (3) if $x, y \in H$, then $x \circ y \subseteq F$ or $(x \circ y) \cap F = \emptyset$.

(4) if $x \in F$ and $H \ni y \geq x$, then $y \in F$.

So a filter of H is a subgroupoid of H satisfying the conditions (2)–(4).

Remark 10. Let H be a \leq -hypergroupoid, F a filter of H and $x, y \in H$. The following are equivalent:

- (1) $x \circ y \subseteq F$ or $(x \circ y) \cap F = \emptyset$.
- (2) if $x \notin F$ or $y \notin F$, then $(x \circ y) \cap F = \emptyset$.

Indeed: (1) \implies (2). Let $x \notin F$ or $y \notin F$. If $x \circ y \subseteq F$ then, since F is a filter, we have $x \in F$ and $y \in F$ which is impossible. Thus we have $x \circ y \not\subseteq F$. Then, by (2), $(x \circ y) \cap F = \emptyset$ and (1) is satisfied.

(2) \implies (1). Let $x \circ y \not\subseteq F$. If $x, y \in F$ then, since F is a filter of H , we have $x \circ y \subseteq F$ which is impossible. Thus we have $x \notin F$ or $y \notin F$. Then, by (2), $(x \circ y) \cap F = \emptyset$, and (1) holds true.

Definition 11. Let H be a \leq -hypergroupoid. A fuzzy subset f of H is called a *fuzzy filter* of H if

- (1) if $x \leq y$ implies $f(x) \leq f(y)$ and
- (2) if $f(x \circ y) = \min\{f(x), f(y)\}$ for every $x, y \in H$

in the sense that if $x, y \in H$ and $u \in x \circ y$, then $f(u) = \min\{f(x), f(y)\}$.

Proposition 12. Let H be a \leq -hypergroupoid. If F is a filter of H , then the fuzzy subset f_F is a fuzzy filter of H . “Conversely”, if F is a nonempty subset of H such that f_F is a fuzzy filter of H , then F is a filter of H .

Proof. \implies . Let $x \leq y$. If $x \notin F$, then $f_F(x) = 0$, so $f_F(x) \leq f_F(y)$. If $x \in F$, then $f_F(x) = 1$. Since $y \in H$ and $y \geq x \in F$, we have $y \in F$. Then $f_F(y) = 1$, and $f_F(x) \leq f_F(y)$.

Let now $x, y \in H$ and $u \in x \circ y$. Then $f_F(u) = \min\{f_F(x), f_F(y)\}$. Indeed:

(a) If $x \circ y \subseteq F$, then $x \in F$ and $y \in F$. Also $u \in F$. Then $f_F(x) = f_F(y) = f_F(u) = 1$, so $f_F(u) = \min\{f_F(x), f_F(y)\}$.

(b) Let $x \circ y \not\subseteq F$. Then $x \notin F$ or $y \notin F$ (since $x, y \in F$ implies $x \circ y \subseteq F$, impossible), then $f_F(x) = 0$ or $f_F(y) = 0$, and $\min\{f_F(x), f_F(y)\} = 0$. On the other hand, since $x \circ y \not\subseteq F$, we have $(x \circ y) \cap F = \emptyset$. Since $u \in x \circ y$, we have $u \notin F$. Then $f_F(u) = 0$, so $f_F(u) = \min\{f_F(x), f_F(y)\}$.

\Leftarrow . Let $x, y \in F$. Then $x \circ y \subseteq F$. Indeed: Let $u \in x \circ y$. By hypothesis, we have $f_F(u) = \min\{f_F(x), f_F(y)\}$. Since $x, y \in F$, we have $f_F(x) = f_F(y) = 1$.

Then $f_F(u) = 1$, and $u \in F$. So F is a subgroupoid of H . Let $x, y \in F$ such that $x \circ y \subseteq F$. Then $x \in F$ and $y \in F$. Indeed: Since $x \circ y \in \mathcal{P}^*(H)$, the set $x \circ y$ is nonempty. Take an element $u \in x \circ y$. Since f_F is a fuzzy filter of H , we have $f_F(u) = \min\{f_F(x), f_F(y)\}$. Suppose $x \notin F$ or $y \notin F$. Then $f_F(x) = 0$ or $f_F(y) = 0$, $\min\{f_F(x), f_F(y)\} = 0$ and $f_F(u) = 0$. On the other hand, since $u \in x \circ y \subseteq F$, we have $f_F(u) = 1$. We get a contradiction. Let $x, y \in H$ such that $x \circ y \not\subseteq F$. Then $(x \circ y) \cap F = \emptyset$. Indeed: Let $u \in (x \circ y) \cap F$. Since $u \in x \circ y$, we have $f_F(u) = \min\{f_F(x), f_F(y)\}$. If $x \notin F$, then $f_F(x) = 0$, then $f_F(u) = 0$. On the other site, since $u \in F$, we have $f_F(u) = 1$ which is impossible, so $x \in F$. In a similar way we prove that $y \in F$, then $(x \circ y) \subseteq F$ which is impossible. Finally, let $x \in F$ and $H \ni y \geq x$. Since f_F is a fuzzy filter of H , we have $1 \geq f_F(y) \geq f_F(x) = 1$, then $f_F(y) = 1$, so $y \in F$. Thus F is a filter of H . \square

In what follows, for a fuzzy subset f of S we introduce the concept of the complement f' of f and prove that f is a fuzzy filter of H if and only if f' is a fuzzy prime ideal of H .

Definition 13. Let H be an hypergroupoid or \leq -hypergroupoid and f a fuzzy subset of H . The fuzzy subset

$$f' : S \rightarrow [0, 1] \text{ defined by } f'(x) = 1 - f(x)$$

is called the *complement* of f (in H).

We remark the following:

- (a) If $x \in H$, then $(f')'(x) = 1 - f'(x) = f(x)$. Thus we have $f'' := (f')' = f$.
- (b) $f(x) \leq f(y) \iff f'(x) \geq f'(y)$ ($x, y \in H$).
- (c) $f(x) = f(y) \iff f'(x) = f'(y)$ ($x, y \in H$).

The Proposition 1 in [3] holds for groupoids and hypergroupoids as well and we have the following lemma.

Lemma 14. Let H be an hypergroupoid, f a fuzzy subset of H and $x, y \in H$. Then we have

$$1 - \min\{f(x), f(y)\} = \max\{f'(x), f'(y)\}.$$

Remark 15. Let H be an hypergroupoid, f a fuzzy subset of H and $x, y \in H$. The following are equivalent:

$$(1) f(x \circ y) = \min\{f(x), f(y)\}.$$

$$(2) f'(x \circ y) = \max\{f'(x), f'(y)\}.$$

Indeed: (1) \implies (2). Let $u \in x \circ y$. By (1), we have $f(u) = \min\{f(x), f(y)\}$. Then, by Lemma 14, we

$$f'(u) = 1 - f(u) = 1 - \min\{f(x), f(y)\} = \max\{f'(x), f'(y)\},$$

and (2) holds true.

(2) \implies (1). Let $u \in x \circ y$. By (2) and Lemma 14, we have

$$f'(u) = \max\{f'(x), f'(y)\} = 1 - \min\{f(x), f(y)\}.$$

Then $f(u) = 1 - f'(u) = \min\{f(x), f(y)\}$, and (1) is satisfied. \square

Definition 16. Let H be a \leq -hypergroupoid. A fuzzy subset f of H is called *fuzzy prime ideal* of H if

$$(1) x \leq y \text{ implies } f(x) \geq f(y) \text{ and}$$

$$(2) f(x \circ y) = \max\{f(x), f(y)\} \text{ for all } x, y \in H$$

that is, if $x, y \in H$ and $u \in x \circ y$, then $f(u) = \max\{f(x), f(y)\}$.

Which means that a fuzzy subset f of H is called a fuzzy prime ideal of H if it is a prime subset of H , that is $f(x \circ y) \leq \max\{f(x), f(y)\}$ for every $x, y \in H$, and at the same time an ideal of H .

Proposition 17. Let H be a \leq -hypergroupoid and f a fuzzy subset of H . Then f is a fuzzy filter of H if and only if the complement f' of f is a fuzzy prime ideal of H .

Proof. \implies . Let $x \leq y$. Since f is a fuzzy filter, we have $f(x) \leq f(y)$, then $f'(x) \geq f'(y)$. Let now $x, y \in H$ and $u \in x \circ y$. Since f is a fuzzy filter, we have $f(u) = \min\{f(x), f(y)\}$. Then, $f'(u) = \max\{f'(x), f'(y)\}$ (cf. also the proof of Remark 15), thus f' is a fuzzy prime ideal of H .

\Leftarrow . Let $x \leq y$. Since f' is a fuzzy ideal of H , we have $f'(x) \geq f'(y)$. Then $f(x) \leq f(y)$. Let now $x, y \in H$ and $u \in x \circ y$. Since f' is a fuzzy prime ideal of H , we have $f'(u) = \max\{f'(x), f'(y)\}$, then $f(u) = \min\{f(x), f(y)\}$. Thus f is a fuzzy filter of H . \square

References

- [1] N. Kehayopulu, On weakly commutative *poe*-semigroups, Semigroup Forum **34**, no. 3 (1987), 367–370.
- [2] N. Kehayopulu, On weakly prime ideals of ordered semigroups, Math. Japon. **35**, no. 6 (1990), 1051–1056.
- [3] N. Kehayopulu, M. Tsingelis, A note of fuzzy sets in semigroups, Sci. Math. **2**, no. 3 (1999), 411–413 (electronic).
- [4] N. Kehayopulu, M. Tsingelis, Fuzzy sets in ordered groupoids, Semigroup Forum **65**, no. 1 (2002), 128–132.

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